Assessing optimality and robustness for the control of dynamical systems

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This work presents a general framework for assessing the quality and robustness of control over a deterministic system described by a state vector $\mathbf{x}(t)$ under external manipulation via a control vector $\mathbf{u}(t)$. The control process is expressed in terms of a cost functional, including the physical objective, penalties, and constraints. The notions of optimality and robustness are expressed in terms of the sign and the magnitude of the cost functional curvature with respect to the controls. Both issues may be assessed from the eigenvalues of the stability operator $\mathcal{S}$ whose kernel $\mathbf{K}(t,\tau)$ is determined by $\overline{\delta\mathbf{u}(t)}/\overline{\delta\mathbf{u}(\tau)}$ for $t_0<t,\ \tau\leq t_f$, where t_0 and t_f are the initial and final times of the control interval. The overbar denotes the constraint that the control satisfies the optimization conditions from minimizing the cost functional. The eigenvalues σ of $\mathcal{S}$ satisfying $\sigma<1$ assure local optimality of a control solution, with $\sigma=1$ being the critical value separating optimal solutions from false solutions (i.e., those with negative second variational curvature of the cost functional). In turn, the maximally robust control solutions with the least sensitivity to field errors also correspond to $\sigma=1$. Thus, sufficiently high sensitivity of the field at one time t to the field at another time τ (i.e., $\sigma>1$) will lead to a loss of local optimality. A simple illustrative example is given from a linear dynamical system, and a bound for the eigenvalue spectrum of the stability operator is presented. The bound is employed to qualitatively analyze control optimality and robustness behavior. A second example of a nonlinear quartic anharmonic oscillator is also presented for stability and robustness analysis. In this case it is proved that the control system kernel is negative definite, implying full stability but only marginal robustness.

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I. INTRODUCTION

A primary concern with any dynamical control application is that the achieved result be suitably optimal and robust [1–8]. Robustness and optimality have been viewed from various perspectives [4,5]. In the context of the present paper, we denote optimality to mean that the cost functional $\mathcal{J}$ is at a positive curvature extremum with respect to the control. Robustness similarly refers to the cost functional being minimally sensitive to disturbances in the control. Thus, in these contexts, the best control solution among multiple possibilities would be the one that minimizes the cost functional while simultaneously having a minimal positive curvature with respect to the control. An analysis of these issues was recently performed for the control of quantum systems [9], and here, we generalize to arbitrary dynamical systems. It will be shown that the dual issues of optimality and robustness are dictated by the spectrum of an integral operator derived from the dynamics. These issues are addressed in the context of control design, assuming that the dynamical system is known. Consideration of the system uncertainty itself has received much attention.

The cost functional $\mathcal{J}$ is standardly composed of objective, penalty, and constraint terms. The dynamical equations for the system are assured to be satisfied by introducing a Lagrange multiplier function, and we will denote $\mathbf{u}(t)$ as the control vector. The physically acceptable solutions correspond to local minima of $\mathcal{J}$, but the first variation criterion $\delta\mathcal{J}/\delta\mathbf{u}(t)=0$ does not guarantee whether the solution is a local minimum or maximum of $\mathcal{J}$. This circumstance can only be assessed by considering the second variation $\delta^2\mathcal{J}/\delta\mathbf{u}(t)\,\delta\mathbf{u}(\tau)$, and determining its positive or negative definite character at each solution obtained from the first variational equations. Even if solutions are determined to be physically acceptable as minima, it is also highly desirable that they be robust to arbitrary incremental variations $\delta\mathbf{u}(t)$ in the control field, as might arise due to errors or uncertainties in the laboratory. In this context, robustness corresponds to a solution associated with minimal positive curvature of the cost functional.

In this work, we show that the eigenvalues of the stability operator $\mathcal{S}$ whose kernel $\mathbf{K}(t,\tau)$ is related to the dynamically constrained (overbar) functional derivative $\overline{\delta\mathbf{u}(t)}/\overline{\delta\mathbf{u}(\tau)}$ for $t_0<t,\tau\leq t_f$ dictate both the optimality and robustness of potential control solutions for manipulating deterministic systems. A formal explicit expression for this operator will be identified for an arbitrary dynamical system. A method for the determination of the spectrum of the stability operator is also given. Some qualitative conclusions will be drawn on the nature of robustness from this bounding relationship. Although the formulation is quite general, a rather simple but meaningful system with a quadratic cost functional and a linear dynamical constraint, and, as a nonlinear system, the classical quartic anharmonic oscillator, are given for illustration.

The paper is organized as follows. Section II presents the formal analysis leading up to the determination of the stability operator. Section III determines the spectrum of the stability operator for a linear dynamical system with a quadratic cost functional. An analysis bounding the spectrum is given,

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followed by two simple illustrative examples to provide insight into the nature of the robustness and optimality. Some brief concluding remarks are presented in Sec. IV.

II. IDENTIFICATION OF THE STABILITY OPERATOR

Consider an optimally controlled deterministic dynamical system whose state is described by the variables $x_i(t)$, $i \leqslant n$, $t \in [t_0, t_f]$ and the system is optimally controlled by the variables $u_j(t)$, $j \leqslant m$, $t \in [t_0, t_f]$. t_0 and t_f prescribe the initial and final times of the optimal control procedure. The optimal control problem is specified by a cost functional composed of objective, penalty, and dynamical constraint terms. The objective term $\mathcal{J}_0$ aims to steer the state towards a target value at the final time. It is explicitly given as

$$\mathcal{J}_0 \equiv \varphi(\mathbf{x}(t_f), t_f). \tag{2.1}$$

Throughout the paper, bold letters will be used to denote vectors or operators. The penalty term $\mathcal{J}_p$ of the cost functional serves to suppress undesirable dynamics or control features (e.g., maintaining the finiteness of the control variables) and is defined through an integral over a Lagrangian function $\mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)$,

$$\mathcal{J}_p \equiv \int_{t_0}^{t_f} dt\, \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{2.2}$$

The dynamics of the system is described by the following differential equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{x}(t_0) = \mathbf{a}, \tag{2.3}$$

where $\mathbf{a}$ and the structure of f are assumed to be known. Finally, a dynamical constraint term $\mathcal{J}_d$ is included in the cost functional to assure that Eq. (2.3) is satisfied through the introduction of a time-dependent Lagrange multiplier vector $\boldsymbol{\lambda}(t)$,

$$\mathcal{J}_d \equiv \int_{t_0}^{t_f} dt\, \boldsymbol{\lambda}^T(t) \left\{ \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \frac{d\mathbf{x}(t)}{dt} \right\}. \tag{2.4}$$

The total cost functional $\mathcal{J}$ is the sum of all of these terms,

$$\mathcal{J} \equiv \mathcal{J}_o + \mathcal{J}_p + \mathcal{J}_d. \tag{2.5}$$

The variables $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\boldsymbol{\lambda}(t)$ are independent quantities, and their nominal values are defined by setting to zero the first variation of the cost functional,

$$\delta \mathcal{J} = 0. \tag{2.6}$$

This relation gives the following equations for the nominal values $\overline{\mathbf{x}}(t)$, $\overline{\mathbf{u}}(t)$, and $\overline{\boldsymbol{\lambda}}(t)$:

$$\frac{d\overline{\mathbf{x}}(t)}{dt} = f(\overline{\mathbf{x}}(t), \overline{\mathbf{u}}(t), t), \tag{2.7a}$$

$$\overline{\mathbf{x}}(t_0) = \mathbf{a}, \tag{2.7b}$$

$$\frac{d\overline{\boldsymbol{\lambda}}(t)}{dt} = -\overline{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)}\right)} - \overline{\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}(t)}\right)^T} \overline{\boldsymbol{\lambda}}(t), \tag{2.8a}$$

$$\overline{\boldsymbol{\lambda}}(t_f) = \overline{\left(\frac{\partial \varphi}{\partial \mathbf{x}(t_f)}\right)}, \tag{2.8b}$$

$$\overline{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}(t)}\right)} + \overline{\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}(t)}\right)^T} \overline{\boldsymbol{\lambda}}(t) = 0, \tag{2.9}$$

where the overbar implies that the nominal values enter the corresponding entity and the arguments of φ, $\mathcal{L}$, and $\mathbf{f}$ are not shown explicitly for simplicity. A concise notation is used for the functional derivatives in the formulas above such as $\partial\mathcal{L}/\partial\mathbf{u}(t)$ denoting a vector whose jth element is $\partial\mathcal{L}/\partial u_j(t)$ while $\partial\mathbf{f}/\partial\mathbf{u}(t)$ denotes a matrix whose jth row and kth column is $\partial f_j/\partial u_k(t)$. These equations suffice to describe the optimally controlled motion of the system. Equations (2.7a) and (2.7b) define a forward evolution from the instant t_0 to t_f by $\overline{\mathbf{x}}(t)$ while Eqs. (2.8a) and (2.8b) describe the backward evolution $\overline{\boldsymbol{\lambda}}(t)$ from t_f to t_0. Both evolutions depend on the control vector $\mathbf{u}(t)$, such that they become compatible at consistent specific values of the control vector $\overline{\mathbf{u}}(t)$ which satisfy Eq. (2.9).

The structure of the last three equations is determined by the dependence of the objective functional φ, the Lagrangian $\mathcal{L}$, and the dynamical function $\mathbf{f}$ on $\mathbf{x}(t)$, $\lambda(t)$, and $\mathbf{u}(t)$. These general nonlinear dependences will likely result in multiple solutions, and this possibility raises the question of which solution will be preferable. Some criteria need to be specified for this purpose, and a natural choice is the desire for robustness with respect to uncertainties or laboratory disturbances in the control vector $\mathbf{u}(t)$. As the control solution is specified by the minimization of the total cost functional $\mathcal{J}$, the best information about the robustness of the control process can be obtained by investigating the second variation of $\mathcal{J}$. If we explicitly write this term $\delta^2\mathcal{J}$ for the nominal values of all entities, then we see that all terms which are composed of the second order variations $\delta^2\mathbf{u}$, $\delta^2\mathbf{x}$, and $\delta^2\boldsymbol{\lambda}$ vanish by virtue of Eqs. (2.7)–(2.9) being valid. Two of the remaining terms are composed of quadratic forms of the first order variations of Eqs. (2.7a), (2.7b) and (2.8a), (2.8b). These equations are not peculiar to the nominal values of the control vector $\overline{\mathbf{u}}(t)$, and they remain valid for any control vector $\mathbf{u}(t)$. Thus, the first order variations of these equations vanish, finally implying that only Eq. (2.9) has a contribution to the second order variations of $\mathcal{J}$. Therefore, we have the following equation for the second variation of the cost functional:

$$\overline{\delta^2\mathcal{J}} = \int_{t_0}^{t_f} dt\, \delta\left[\frac{\partial \mathcal{L}}{\partial \mathbf{u}(t)} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}(t)}\right)^T \boldsymbol{\lambda}(t)\right]^T \delta\mathbf{u}(t), \tag{2.10}$$

which can be written more explicitly as follows:

$$\begin{aligned}\overline{\delta^2\mathcal{J}} = \int_{t_0}^{t_f} dt \{ &\delta\mathbf{x}(t)^T [\mathbf{L}_{xu}(t) + \mathbf{F}_{xu}(t)] \delta\mathbf{u}(t) \\ &+ \delta\mathbf{u}(t)^T [\mathbf{L}_{uu}(t) + \mathbf{F}_{uu}(t)] \delta\mathbf{u}(t) \\ &+ \delta\boldsymbol{\lambda}(t)^T \mathbf{F}_u(t) \delta\mathbf{u}(t)\},\end{aligned} \tag{2.11}$$

where

$$\mathbf{F}_u(t) \equiv \frac{\partial \mathbf{f}}{\partial \mathbf{u}(t)}, \tag{2.12a}$$

$$\mathbf{F}_{xu}(t) \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial^2 f_j}{\partial \mathbf{x}(t) \partial \mathbf{u}(t)}, \tag{2.12b}$$

$$\mathbf{F}_{uu}(t) \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial^2 f_j}{\partial \mathbf{u}(t)^2}, \tag{2.12c}$$

and

$$\mathbf{L}_{xu}(t) \equiv \frac{\partial^2 \mathcal{L}}{\partial \mathbf{x}(t) \partial \mathbf{u}(t)}, \tag{2.13a}$$

$$\mathbf{L}_{uu}(t) \equiv \frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}(t)^2}. \tag{2.13b}$$

The notation in Eqs. (2.12a)–(2.12c) and Eqs. (2.13a) and (2.13b) is consistent with that used previously, such that $\partial^2 \mathcal{L}/\partial \mathbf{x}(t) \partial \mathbf{u}(t)$ denotes a matrix whose jth row and kth column is $\partial^2 \mathcal{L}/\partial x_j(t) \partial u_k(t)$. In this notation the order of the differentiation is important, which results in the transposition of the matrix,

$$\left[\frac{\partial^2 \mathcal{L}}{\partial \mathbf{x}(t) \partial \mathbf{u}(t)} \right]^T \equiv \frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}(t) \partial \mathbf{x}(t)}. \tag{2.14}$$

The second variation of the cost functional given in Eq. (2.11) is a quadratic form over the first variations of $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\lambda(t)$. Since $\delta\mathbf{x}(t)$ and $\delta\boldsymbol{\lambda}(t)$ depend on $\delta\mathbf{u}(t)$ (i.e., a laboratory variation of the control), we can write the following relations in terms of their sensitivity coefficients with respect to the control vector:

$$\delta\mathbf{x}(t) = \int_{t_0}^{t_f} d\tau\, \mathbf{S}_x(t,\tau)\, \delta\mathbf{u}(\tau), \tag{2.15a}$$

$$\delta\boldsymbol{\lambda}(t) = \int_{t_0}^{t_f} d\tau\, \mathbf{S}_\lambda(t,\tau)\, \delta\mathbf{u}(\tau). \tag{2.15b}$$

A similar relation also can be written for the control vector itself as follows:

$$\delta\mathbf{u}(t) = \int_{t_0}^{t_f} d\tau\, \delta(t-\tau)\, \delta\mathbf{u}(\tau). \tag{2.16}$$

These relations lead to

$$\overline{\delta^2 \mathcal{J}} = \int_{t_0}^{t_f} dt \int_{t_0}^{t_f} d\tau\, \delta\mathbf{u}(\tau)^T \{ \mathbf{S}_x(t,\tau)^T [\mathbf{L}_{xu}(t) + \mathbf{F}_{xu}(t)] + \mathbf{S}_\lambda(t,\tau)^T \mathbf{F}_u(t) + \delta(t-\tau)[\mathbf{L}_{uu}(t) + \mathbf{F}_{uu}(t)] \} \delta\mathbf{u}(t). \tag{2.17}$$

If we now assume that the matrix sum which postmultiplies the Dirac δ function in Eq. (2.17) is positive definite, then we can define a symmetric weight matrix $\mathbf{W}(t)$ as follows:

$$\mathbf{W}(t)^2 \equiv \mathbf{L}_{uu}(t) + \mathbf{F}_{uu}(t). \tag{2.18}$$

This positive definite assumption is valid for the common case that $\mathbf{L}$ is quadratic in $\mathbf{u}$ and $\mathbf{F}$ is linear in $\mathbf{u}$ such that $\mathbf{F}_{uu} = \mathbf{0}$. We can define a unit operator I acting on an arbitrary integrable vector function $\mathbf{g}(t)$ over the interval $t \in [t_0, t_f]$ as

$$\mathcal{I}\mathbf{g}(t) \equiv \int_{t_0}^{t_f} d\tau\, \delta(t-\tau)\mathbf{g}(\tau) \tag{2.19}$$

and a symmetric operator $\mathcal{S}$,

$$\mathcal{S}\mathbf{g}(t) \equiv \int_{t_0}^{t_f} dt\, \mathbf{K}(t,\tau)\mathbf{g}(\tau) \tag{2.20}$$

in terms of a symmetrized kernel matrix $\mathbf{K}(t,\tau)$,

$$\mathbf{K}(t,\tau) \equiv -\tfrac{1}{2}\mathbf{W}(t)^{-1} \{ \mathbf{S}_x(t,\tau)^T [\mathbf{L}_{xu}(t) + \mathbf{F}_{xu}(t)] + [\mathbf{L}_{ux}(\tau) + \mathbf{F}_{ux}(\tau)] \mathbf{S}_x(\tau,t) + \mathbf{S}_\lambda(t,\tau)^T \mathbf{F}_u(t) + \mathbf{F}_u(\tau)^T \mathbf{S}_\lambda(\tau,t) \} \mathbf{W}(\tau)^{-1}. \tag{2.21}$$

Since Eq. (2.17) does not change when t and τ are interchanged we can conclude the following concise form for the second variation of the cost functional:

$$\overline{\delta^2 \mathcal{J}} = \int_{t_0}^{t_f} dt\, \delta\mathbf{u}(t)^T \mathbf{W}(t) [\mathcal{I} - \mathcal{S}] \mathbf{W}(t)\, \delta\mathbf{u}(t). \tag{2.22}$$

The kernel $\mathbf{K}(t,\tau)$ given by Eq. (2.21) includes $\mathbf{u}$ as an arbitrary function. However, we want to confine ourselves to the case of optimal solutions because the robustness and optimality analysis are important for only these solutions. Hence, we will take the following equation as the definition of the kernel:

$$\mathbf{K}(t,\tau) \equiv -\tfrac{1}{2} \overline{\mathbf{W}}(t)^{-1} \{ \overline{\mathbf{S}}_x(t,\tau)^T [\overline{\mathbf{L}}_{xu}(t) + \overline{\mathbf{F}}_{xu}(t)] + [\overline{\mathbf{L}}_{ux}(\tau) + \overline{\mathbf{F}}_{ux}(\tau)] \overline{\mathbf{S}}_x(\tau,t) + \overline{\mathbf{S}}_\lambda(t,\tau)^T \overline{\mathbf{F}}_u(t) + \overline{\mathbf{F}}_u(\tau)^T \overline{\mathbf{S}}_\lambda(\tau,t) \} \overline{\mathbf{W}}(\tau)^{-1}. \tag{2.23}$$

The kernel $\mathbf{K}(t,\tau)$ is closely related to the functional derivative of $\mathbf{u}(t)$ with respect to its value at another time $\mathbf{u}(\tau)$, when $\mathbf{u}(t)$ is confined to the set of solutions to Eq. (2.9). If we consider the set of variations $\delta\mathbf{u}(t)$ such that $\mathbf{u}(t)$ and $\mathbf{u}(t) + \delta\mathbf{u}(t)$ remain in the set of solutions to Eq. (2.9), then we can functionally differentiate Eq. (2.9) to obtain

$$\overline{\delta\mathbf{x}}(t)^T [\overline{\mathbf{L}}_{xu}(t) + \overline{\mathbf{F}}_{xu}(t)] \overline{\delta\mathbf{u}}(t) + \overline{\delta\mathbf{u}}(t)^T [\overline{\mathbf{L}}_{uu}(t) + \overline{\mathbf{F}}_{uu}(t)] \overline{\delta\mathbf{u}}(t) + \overline{\delta\boldsymbol{\lambda}}(t)^T \overline{\mathbf{F}}_u(t) \overline{\delta\mathbf{u}}(t) = 0, \tag{2.24}$$

where all overbarred quantities are evaluated at the nominal values of the state and control variables. Equations (2.15a) and (2.15b) can be quickly rewritten for the overbarred entities as

$$\overline{\delta\mathbf{x}}(t)=\int_{t_0}^{t_f}d\tau\,\overline{\mathbf{S}}_x(t,\tau)\,\delta\mathbf{u}(\tau), \tag{2.25a}$$

$$\overline{\delta\boldsymbol{\lambda}}(t)=\int_{t_0}^{t_f}d\tau\,\overline{\mathbf{S}}_\lambda(t,\tau)\,\delta\mathbf{u}(\tau). \tag{2.25b}$$

The interpretation of $\delta\mathbf{u}(t)^T$ in Eq. (2.24) calls for care, as it is the incremental response of the field about the nominal solution to an *arbitrary* variation $\delta\mathbf{u}(t)$, such that

$$\overline{\delta\mathbf{u}}(t)=\int_{t_0}^{t_f}d\tau\overline{\left(\frac{\delta\mathbf{u}(t)}{\delta\mathbf{u}(\tau)}\right)}\delta\mathbf{u}(\tau). \tag{2.26}$$

The use of Eqs. (2.25a), (2.25b), and (2.26) in Eq. (2.24) yields

$$\int_{t_0}^{t_f}d\tau\,\delta\mathbf{u}(\tau)^T\Bigg\{\overline{\mathbf{S}}_x(t,\tau)[\overline{\mathbf{L}}_{xu}(t)+\overline{\mathbf{F}}_{xu}(t)] +\overline{\left(\frac{\delta\mathbf{u}(t)}{\delta\mathbf{u}(\tau)}\right)}^T[\overline{\mathbf{L}}_{uu}(t)+\overline{\mathbf{F}}_{uu}(t)]+\overline{S_\lambda}(t)^T\overline{\mathbf{F}}_u(t)\Bigg\}\overline{\delta\mathbf{u}}(t) =0, \tag{2.27}$$

where we have employed the symmetry property of $\overline{\mathbf{L}}_{uu}(t)$ and $\overline{\mathbf{F}}_{uu}(t)$. Equation (2.27) leads to the relation

$$\overline{\left(\frac{\delta\mathbf{u}(t)}{\delta\mathbf{u}(\tau)}\right)}^T=-[\overline{\mathbf{L}}_{uu}(t)+\overline{\mathbf{F}}_{uu}(t)]^{-1}\{\overline{\mathbf{S}}_x(t,\tau)[\overline{\mathbf{L}}_{xu}(t)+\overline{\mathbf{F}}_{xu}(t)] +\overline{S_\lambda}(t)^T\overline{\mathbf{F}}_u(t)\}. \tag{2.28}$$

A careful investigation shows that the kernel can be identified as

$$\mathbf{K}(t,\tau)=\frac{1}{2}\left[\mathbf{W}(t)\overline{\left(\frac{\delta\mathbf{u}(t)}{\delta\mathbf{u}(\tau)}\right)}^T+\overline{\left(\frac{\delta\mathbf{u}(\tau)}{\delta\mathbf{u}(t)}\right)}\mathbf{W}(\tau)\right], \tag{2.29}$$

which is the symmetric part of $\mathbf{W}(t)[\overline{\delta\mathbf{u}(t)}/\overline{\delta\mathbf{u}(\tau)}]^T$ upon the $t\leftrightarrow\tau$ interchange.

The operator $\mathcal{S}$ is responsible for characterizing the optimality and robustness of the control solutions. Since it is symmetric by construction, its spectrum is real [10]. As long as its largest eigenvalue about a nominal solution does not exceed 1, then the corresponding solution is acceptable; otherwise, the optimal solution will correspond to local maxima of the cost functional. The distance between 1 and the largest eigenvalue of $\mathcal{S}$ determines the degree of robustness. A smaller distance corresponds to a smaller curvature so as to make the solution less sensitive to the changes in the control. The determination of the spectrum of the operator $\mathbf{S}$ for an arbitrary system will call for standard numerical methods, and little of a general nature can be said before the analysis. Hence, we will discuss the spectral evaluation on a simple illustrative system in the next section.

Before closing this section, we can make a few qualitative comments on the norm of the operator kernel $\mathbf{K}$. According to Eq. (2.23), increasing the magnitude of $\overline{\mathbf{W}}(t)$ will diminish the norm of the kernel, if the other terms vary comparatively slowly. This is possible, for example, when $\overline{\mathbf{L}}_{uu}(t)$ is large while all other terms vary slowly. We may interpret $\overline{\mathbf{L}}_{uu}(t)$ as the weight of the field fluence. This means that the norm of the kernel decreases as the contribution of the field fluence penalty term increases. Similarly, Eq. (2.23) implies that the increasing sensitivity of the state and the Lagrange multiplier vectors, $\overline{\mathbf{S}}_x(t)$ and $\overline{\mathbf{S}}_\lambda(t)$, to the control variable raises the value of the kernel's norm.

III. ILLUSTRATIONS

This section will apply the general optimality and robustness formalism above to two examples. The first is a linear system with a single control and a quadratic cost functional. Bounds will be deduced for the eigenvalues of the operator $\mathbf{S}$, and a simple analytically soluble special case will be examined to attain further insight. The last subsection (Sec. III B) gives the second example on the stability and robustness of a quartic anharmonic oscillator.

A. Linear systems

In Secs. III A 1–III A 3, we will present the general spectral analysis for a linear system with a quadratic cost functional. The treatment will include a spectral analysis and bounds for the kernel along with their relationships to general optimality and robustness properties. Finally an application will be given for a simple analytically soluble linear system.

1. Spectral analysis

Consider an optimally controlled system with an n-dimensional state vector $\mathbf{x}(t)$ and a single control function $u(t)$ associated with a quadratic cost functional. The Lagrangian has the form

$$\mathcal{L}(\mathbf{x}(t),\mathbf{u}(t),t)\equiv\tfrac{1}{2}\mathbf{x}(t)^T\mathbf{Q}_x(t)\mathbf{x}(t)+\tfrac{1}{2}\Omega_u(t)u(t)^2,$$
$$\Omega_u(t)>0,\quad \mathbf{Q}_x(t)^T=\mathbf{Q}_x(t),\quad t\in[t_0,t_f] \tag{3.1}$$

and $\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)$ is given in terms of $\mathbf{A}(t)$ and $\mathbf{b}(t)$,

$$\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)\equiv\mathbf{A}(t)\mathbf{x}(t)+\mathbf{b}(t)u(t). \tag{3.2}$$

The objective term has the form

$$\varphi(\mathbf{x}(t_f),t_f)\equiv\tfrac{1}{2}[\mathbf{x}(t_f)-\tilde{\mathbf{x}}]^T\mathbf{Q}_\varphi(t_f)[\mathbf{x}(t_f)-\tilde{\mathbf{x}}],$$
$$\mathbf{Q}_\varphi(t_f)^T=\mathbf{Q}_\varphi(t_f) \tag{3.3}$$

where $\tilde{\mathbf{x}}$ is the prescribed target and $\mathbf{Q}_\varphi(t_f)$ is assumed to be positive definite.

The nominal values $\overline{\mathbf{x}}(t),\overline{\boldsymbol{\lambda}}(t),\overline{u}(t)$ satisfy the equations

$$\frac{d\overline{\mathbf{x}}}{dt}=\mathbf{A}(t)\overline{\mathbf{x}}(t)+\mathbf{b}(t)\overline{u}(t), \tag{3.4a}$$

$$\overline{\mathbf{x}}(t_0)=\mathbf{a}, \tag{3.4b}$$

$$\frac{d\overline{\boldsymbol{\lambda}}(t)}{dt}=-\mathbf{Q}_x(t)\overline{\mathbf{x}}(t)-\mathbf{A}(t)^T\overline{\boldsymbol{\lambda}}(t), \tag{3.5a}$$

$$\overline{\boldsymbol{\lambda}}(t_f)=\mathbf{Q}_\varphi(t_f)[\overline{\mathbf{x}}(t_f)-\tilde{\mathbf{x}}], \tag{3.5b}$$

$$\Omega_u(t)\overline{u}(t)+\mathbf{b}(t)^T\overline{\boldsymbol{\lambda}}(t)=0, \tag{3.6}$$

where we have used

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}(t)}=\mathbf{A}(t), \tag{3.7a}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)}=\mathbf{Q}_x(t)\mathbf{x}(t), \tag{3.7b}$$

$$\frac{\partial \varphi}{\partial \mathbf{x}(t_f)}=\mathbf{Q}_\varphi(t_f)[\mathbf{x}(t_f)-\tilde{\mathbf{x}}], \tag{3.7c}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}(t)}=\mathbf{b}(t), \tag{3.7d}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}(t)}=\Omega_u(t)u(t). \tag{3.7e}$$

The solution for $\overline{\mathbf{x}}(t)$ can be obtained in terms of $u(t)$ through standard analytic techniques to give the following explicit expression:

$$\overline{\mathbf{x}}(t)=\mathbf{P}(t)\mathbf{a}+\int_{t_0}^{t}d\tau\,\mathbf{P}(t)\mathbf{P}(\tau)^{-1}\mathbf{b}(\tau)u(\tau), \tag{3.8}$$

where $\mathbf{P}(t)$ satisfies

$$\frac{d\mathbf{P}(t)}{dt}=\mathbf{A}(t)\mathbf{P}(t), \tag{3.9a}$$

$$\mathbf{P}(t_0)=\mathbf{I}. \tag{3.9b}$$

A similar expression can be obtained to relate $\overline{\boldsymbol{\lambda}}(t)$ to $\overline{\mathbf{x}}(t)$ as follows:

$$\overline{\boldsymbol{\lambda}}(t)=\mathbf{P}(t)^{-1^T}\mathbf{P}(t_f)^T\mathbf{Q}_\varphi(t_f)[\overline{\mathbf{x}}(t_f)-\tilde{\mathbf{x}}]+\int_{t}^{t_f}d\tau\,\mathbf{P}(t)^{-1^T}\mathbf{P}(\tau)^T\mathbf{Q}_x(\tau)\overline{\mathbf{x}}(\tau), \tag{3.10}$$

where we have used the fact

$$\frac{d\mathbf{P}(t)^{-1}}{dt}=-\mathbf{P}(t)^{-1}\mathbf{A}(t), \tag{3.11a}$$

$$\mathbf{P}(t_0)^{-1}=\mathbf{I}. \tag{3.11b}$$

Equations (3.8) and (3.10) suffice to specify the kernel $\mathbf{K}$ of the operator $\mathbf{S}$, since we can identify the specific values

$$\mathbf{F}_u=\mathbf{b}(t), \tag{3.12a}$$

$$\mathbf{F}_{uu}=0, \tag{3.12b}$$

$$\mathbf{F}_{xu}=0, \tag{3.12c}$$

$$\mathbf{L}_{uu}=\Omega_u(t), \tag{3.13a}$$

$$\mathbf{L}_{xu}=0, \tag{3.13b}$$

which imply that

$$\mathbf{W}(t)=\Omega_u(t)^{1/2} \tag{3.14}$$

and from Eq. (2.21),

$$\mathbf{K}(t,\tau)=\Omega_u(t)^{-1/2}[\overline{\mathbf{S}}_\lambda(t,\tau)^T\mathbf{b}+\mathbf{b}^T\overline{\mathbf{S}}_\lambda(\tau,t)]\Omega_u(\tau)^{-1/2}. \tag{3.15}$$

Equation (3.15) shows that the explicit determination of the kernel necessitates the evaluation of the sensitivity coefficient $\overline{\mathbf{S}}_\lambda(t,\tau)$. To this end we can start with the determination of $\overline{\mathbf{S}}_x(t,\tau)$. This can be easily done by functionally differentiating Eq. (3.8), with respect to $u(t)$. This gives

$$\overline{\mathbf{S}}_x(t,\tau)=\begin{cases}0, & t<\tau\\ \mathbf{P}(t)\mathbf{P}(\tau)^{-1}\mathbf{b}(\tau), & t\geqslant\tau,\end{cases} \tag{3.16}$$

A similar treatment of Eq. (3.10) gives the following relation between $\overline{\mathbf{S}}_\lambda(t,\tau)$ and $\overline{\mathbf{S}}_x(t,\tau)$:

$$\overline{\mathbf{S}}_\lambda(t,\tau)=\mathbf{P}(t)^{-1^T}\mathbf{P}(t_f)^T\mathbf{Q}_\varphi(t_f)\overline{\mathbf{S}}_x(t_f,\tau)+\int_t^{t_f}dt_1\,\mathbf{P}(t)^{-1^T}\mathbf{P}(t_1)^T\mathbf{Q}_x(t_1)\overline{\mathbf{S}}_x(t_1,\tau). \tag{3.17}$$

The use of Eq. (3.16) in Eq. (3.17) yields

$$\overline{\mathbf{S}}_\lambda(t,\tau)=\mathbf{P}(t)^{-1^T}\mathbf{P}(t_f)^T\mathbf{Q}_\varphi(t_f)\mathbf{P}(t_f)\mathbf{P}(\tau)^{-1}\mathbf{b}(\tau)+\int_{\beta(t,\tau)}^{t_f}dt_1\,\mathbf{P}(t)^{-1^T}\mathbf{P}^T(t_1)\mathbf{Q}_x(t_1)\mathbf{P}(t_1)\mathbf{P}(\tau)^{-1}\mathbf{b}(\tau), \tag{3.18}$$

where

$$\beta(t,\tau)=\begin{cases}\tau, & t\leqslant\tau\\ \tau, & t\geqslant\tau.\end{cases} \tag{3.19}$$

Since $\overline{\mathbf{S}}_\lambda(t,\tau)$ and $\mathbf{b}(t)$ are n-dimensional vector functions, the kernel of the stability operator, $\mathbf{K}(t,\tau)$, is a scalar. Therefore, we can write

$$\mathbf{K}(t,\tau)=\Omega_u(t)^{-1/2}[K_1(t,\tau)+K_1(\tau,t)]\Omega_u(\tau)^{=1/2}, \tag{3.20}$$

$$K_1(t,\tau)=\sum_{j=1}^{n}\kappa_j^{(\varphi)}(t_f)\,\eta_j(t)\,\eta_j(\tau)+K_2(t,\tau), \tag{3.21}$$

$$K_2(t,\tau)=\int_{\beta(t,\tau)}^{t_f}dt_1\,\mathbf{b}^T(t)\mathbf{P}(t)^{-1^T}\mathbf{P}(t_1)^T\mathbf{Q}_x(t)\times\mathbf{P}(t_1)\mathbf{P}(\tau)^{-1}\mathbf{b}(\tau), \tag{3.22}$$

$$\eta_j(t)=\Omega_u(t)^{-1/2}\mathbf{q}_j^{(\varphi)}(t_f)^T\mathbf{P}(t_f)\mathbf{P}(t)^{-1}\mathbf{b}(t),\quad j=1,2,3,\ldots,n. \tag{3.23}$$

In these equations, $\kappa_j^{(\varphi)}(t_f)$, $\mathbf{q}_j^{(\varphi)}(t_f)$ denote the jth eigenvalue and the corresponding normalized eigenvector of the matrix $\mathbf{Q}_\varphi(t_f)$. The eigenvalues are real since $\mathbf{Q}_\varphi(t_f)$ is symmetric.

Now we can formulate the eigenvalue problem of the operators as

$$\mathcal{S}\xi(t)=\sigma\xi(t), \tag{3.24}$$

or more explicitly

$$\mathcal{S}_1\xi(t)+\sum_{j=1}^{n}\nu_j\eta_j(t)=\sigma\xi(t), \tag{3.25}$$

where the integral operator $\mathcal{S}_1$ is defined through an arbitrary vector function $\mathbf{g}(t)$ which is integrable over the interval $t\in[t_0,t_f]$ as

$$\mathcal{S}_1\mathbf{g}(t)\equiv\int_{t_0}^{t_f}d\tau\ \Omega_u(t)^{-1/2}[K_2(t,\tau)+K_2(\tau,t)]\times\Omega_u(\tau)^{-1/2}\mathbf{g}(\tau). \tag{3.26}$$

ν_j, $j=1,2,\ldots,n$ is defined as follows:

$$\nu_j\equiv\int_{t_0}^{t_f}dt\,\eta_j(t)\xi(t). \tag{3.27}$$

To reduce the above eigenvalue problem to an algebraic equation for σ we can formally write

$$\xi(t)=\sum_{k=1}^{n}\nu_k[\sigma\mathcal{I}-\mathcal{S}_1]^{-1}\eta_k(t). \tag{3.28}$$

Premultiplication of Eq. (3.28) by $\eta_j(t)$ and integration over t between t_0 and t_f gives the equations

$$\sum_{k=1}^{n}\rho_{jk}(\sigma)\nu_k=\nu_j,\quad j=1,2,\ldots,n \tag{3.29}$$

where

$$\rho_{jk}(\sigma)\equiv\int_{t_0}^{t_f}dt\,\eta_j(t)[\sigma\mathcal{I}-\mathcal{S}_1]^{-1}\eta_k(t),\quad j,k=1,2,\ldots,n. \tag{3.30}$$

If we let $\boldsymbol{\rho}(\sigma)$ denote the n-dimensional matrix whose elements are represented by $\rho_{jk}(\sigma)$, then we need to evaluate the possible σ values which make at least one eigenvalue of $\boldsymbol{\rho}$ unity. These values form a set for the solution of the eigenproblem. Their evaluation also produces the values of the ν_j, $j=1,2,\ldots,n$ parameters within an arbitrary multiplicative constant. After having these values the remaining procedure is just a matter of evaluation of $\xi(t)$ from Eq. (3.28) within the same arbitrary multiplicative constant. This procedure needs an explicit expression of $\rho_{jk}(\sigma)$ in terms of σ for all values of j and k. To this end we can use a number of means, including a series expansion, Stieltjes series [9], characteristic function approach [10,11], Padé approximants [9], etc. To go further calls for numerical calculations, and for the purposes of this paper, it suffices to place bounds on the spectrum of S, as presented in the next subsection.

2. Spectral bounds and general optimality and robustness properties

We seek simple expressions as bounds for the spectrum of $\mathbf{S}$. To this end, the following bounds $\mathcal{B}(\,)$ for the norms of various quantities are useful:

$$\mathcal{B}(\mathbf{b})\equiv\left(\int_{t_0}^{t_f}dt\,\mathbf{b}(t)^T\mathbf{b}(t)\right)^{1/2}, \tag{3.31}$$

$$\mathcal{B}(\mathbf{P})\equiv\left\{\frac{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\mathbf{P}(t)^T\mathbf{P}(t)\boldsymbol{\zeta}(t)}{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\boldsymbol{\zeta}(t)}\right\}_{\max}^{1/2}, \tag{3.32}$$

$$\mathcal{B}(\mathbf{P}^{-1})\equiv\left\{\frac{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\mathbf{P}(t)^{-1^T}\mathbf{P}(t)^{-1}\boldsymbol{\zeta}(t)}{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\boldsymbol{\zeta}(t)}\right\}_{\max}^{1/2}, \tag{3.33}$$

$$\mathcal{B}(\mathbf{Q}_x)\equiv\left\{\frac{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\mathbf{Q}_x(t)^T\mathbf{Q}_x(t)\boldsymbol{\zeta}(t)}{\int_{t_0}^{t_f}dt\,\boldsymbol{\zeta}(t)^T\boldsymbol{\zeta}(t)}\right\}_{\max}^{1/2}, \tag{3.34}$$

where the subscript max means the maximization with respect to $\boldsymbol{\zeta}$. These definitions, combined with Eqs. (3.15)–(3.18), lead to

$$\mathcal{B}(\mathcal{S})=[\mathcal{B}(\mathbf{Q}_\varphi)+\mathcal{B}(\mathbf{Q}_x)]\mathcal{B}(\mathbf{P})^2\mathcal{B}(\mathbf{P}^{-1})^2\mathcal{B}(\mathbf{b})^2, \tag{3.35}$$

where $\mathcal{B}(\mathbf{Q}_\varphi)$ is the largest one of the $\kappa_j^{(\varphi)}(t_f)$ parameters in Eq. (3.21). If we define the condition number K_p for the propagator $\mathbf{P}$ as

$$K_p\equiv\mathcal{B}(\mathbf{P})\mathcal{B}(\mathbf{P}^{-1}), \tag{3.36}$$

then Eq. (3.35) becomes

$$\mathcal{B}(\mathcal{S})=[\mathcal{B}(\mathbf{Q}_\varphi)+\mathcal{B}(\mathbf{Q}_x)]K_p^2\mathcal{B}(\mathbf{b})^2. \tag{3.37}$$

If the bound given above for the operator remains smaller than 1, then the control of the system under consideration is stable. However, the bounds which exceed 1 do not mean that the control of the system is unstable due to the fact that the norm analysis might be too conservative. Thus, $\mathcal{B}(\mathcal{S})<1$ is not a necessary condition for optimality. In the case of acceptable optimal control solutions, the difference between $\mathcal{B}(\mathcal{S})$ and 1 gives a measure of the robustness of the solutions. Smaller differences imply more robustness in the control solutions since the reduced curvature of the cost functional leaves the system less sensitive to changes in the control $\mathbf{u}(t)$.

Some qualitative conclusions may be drawn from the structure of Eq. (3.37). To do so we will assume $\mathcal{B}(\mathcal{S})<1$ and the issue of interest is how the physical variables act to increase the robustness by $\mathcal{B}(\mathcal{S})\rightarrow 1$; an extension of this behavior eventually results in an unacceptable physical solution $\mathcal{B}(\mathcal{S})>1$ with $\overline{\delta^2\mathcal{J}}<0$ With these comments in mind we may draw the following conclusions from Eq. (3.37).

(1) Objective. Increasing the contribution of $\mathcal{B}(\mathbf{Q}_\varphi)$ from the objective will enhance the robustness.

(2) Penalty. Increasing the contribution of $\mathcal{B}(\mathbf{Q}_x)$ from the penalty will enhance the robustness.

(3) Control coupling vector. Enhanced robustness occurs with increasing magnitude of the control coupling vector $\mathbf{b}$. This, apparently, arises due to more effective control regardless of the control strength.

(4) Propagator's condition number. As the condition number K_p increases, a corresponding enhancement of robustness occurs.

Once again, all of the circumstances in points (1)–(4) (especially the later one), when taken beyond a critical limit, lead to a nonoptimal control solution. Additional subtleties might arise from a full analysis of the eigenvalues of the operator $\mathcal{S}$, beyond the simple bounding behavior examined here. This point is amplified by observing that the bound given above does not explicitly depend on the nominal values of the state and control variables. This is due to the specific structure of the example above. If the dynamical constraint is chosen with a nonlinear dependence on these variables, or if the cost functional is chosen to have a higher order nonlinearity than a quadratic dependence on the variables, then the nominal values will appear in the bound.

3. Application to an analytically soluble linear system

Consider a simple nontrivial system such that $\mathbf{b}$ is a constant n-dimensional vector, $\mathbf{A}$ is an n-dimensional symmetric constant square matrix, and $\mathbf{Q}_x(t)$ vanishes. The objective term $\mathbf{Q}_\varphi(t)$ is assumed to be a projection operator

$$\mathbf{Q}_\varphi(t) \equiv \mathbf{v}_1\mathbf{v}_1^T, \tag{3.38}$$

where $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$. Thus, the goal is to steer the system state towards the eigenvector $\mathbf{v}_1$ of $\mathbf{A}$. We also assume that

$$\Omega_u(t) \equiv 1. \tag{3.39}$$

For this system, the kernel of the operator $\mathcal{S}$ can be explicitly written as

$$\mathbf{K}(t,\tau) = (\mathbf{b}^T\mathbf{v}_1)^2 e^{\nu_1(2t_f - t - \tau)}. \tag{3.40}$$

The operator $\mathcal{S}$ has only one eigenvalue, given by

$$\sigma_1 = (\mathbf{b}^T\mathbf{v}_1)^2 \frac{e^{2\nu_1(t_f - t_0)} - 1}{2\nu_1}. \tag{3.41}$$

This value increases unboundedly if $\nu_1 > 0$ when the control time $(t_f - t_0)$ tends to go to infinity. This means that the solution will be lost after a specific value of the control time interval. This is because positive ν_1 corresponds to a dynamically unstable system. The case where $\nu_1 = 0$ creates a similar chance of losing the solution when the control time increases beyond a specific value. In the case of negative ν_1 values, the σ_1 value is bounded from above by $(\mathbf{b}^T\mathbf{v}_1)^2$ for all control times. Hence, the solution of the optimal control problem will exist if $(\mathbf{b}^T\mathbf{v}_1)^2 < 1$. In the case where $(\mathbf{b}^T\mathbf{v}_1)^2 > 1$ and $\nu_1 = 0$, one needs to use a control time interval smaller than a specific value to get positive curvature in the cost functional and therefore optimality in the solution.

B. An application to a quartic anharmonic oscillator

Consider an optimally controlled classical quartic anharmonic oscillator whose Hamiltonian is

$$H \equiv \frac{1}{2m}x_2(t)^2 + \frac{k_1}{2}x_1(t)^2 + \frac{k_2}{4}x_1(t)^4 - bx_1(t)u(t), \tag{3.42}$$

where k_1, k_2, and b are constants, m is the mass, $x_1(t)$ is the particle position, and $x_2(t)$ is the associated momentum.

The Lagrangian in Eq. (2.2) is specified as

$$\mathcal{L}(\overline{\mathbf{x}}(t), \overline{\mathbf{u}}(t), t) \equiv \tfrac{1}{2}\overline{\mathbf{x}}(t)^T\mathbf{Q}\overline{\mathbf{x}}(t) + \tfrac{1}{2}\omega\overline{u}(t)^2,$$

$$\omega > 0, \quad \mathbf{Q}^T = \mathbf{Q}, \quad t \in [0, t_f]. \tag{3.43}$$

The objective term in the cost functional of Eq. (2.1) is given as

$$\varphi(\overline{\mathbf{x}}(t_f), t_f) \equiv \tfrac{1}{2}[\overline{\mathbf{x}}(t_f) - \tilde{\mathbf{x}}]^T\mathbf{R}[\overline{\mathbf{x}}(t_f) - \tilde{\mathbf{x}}], \tag{3.44}$$

with $\mathbf{R}^T = \mathbf{R}$ being positive definite. The equations of motion in Eq. (2.3) for this case are

$$\frac{d\overline{x_1}(t)}{dt} = \frac{1}{m}\overline{x_2}(t), \tag{3.45a}$$

$$\frac{d\overline{x_2}(t)}{dt} = -k_1\overline{x_1}(t) - k_2\overline{x_1}(t)^3 + b\overline{u}(t), \tag{3.45b}$$

$$\overline{x_1}(0) = a_1, \tag{3.45c}$$

$$\overline{x_2}(0) = a_2, \tag{3.45d}$$

with a_1 and a_2 being the initial position and momentum. Similarly, the Lagrange multiplier Eqs. (2.8) and (2.9) become

$$\frac{d\overline{\lambda_1}(t)}{dt} = -Q_{11}\overline{x_1}(t) - Q_{12}\overline{x_2}(t) + [k_1 + 3k_2\overline{x_1}(t)^2]\overline{\lambda_2}(t), \tag{3.46a}$$

$$\frac{d\overline{\lambda_2}(t)}{dt} = -Q_{12}\overline{x_1}(t) - Q_{22}\overline{x_2}(t) - \frac{1}{m}\overline{\lambda_1}(t), \tag{3.46b}$$

$$\overline{\boldsymbol{\lambda}}(t_f) = \mathbf{R}[\overline{\mathbf{x}}(t_f) - \tilde{\mathbf{x}}], \tag{3.46c}$$

$$\omega\overline{u}(t) + b\overline{\lambda_2}(t) = 0. \tag{3.47}$$

The following changes to dimensionless variables are made to facilitate the subsequent analysis: $t \to \sqrt{m/k_1}\,t$, $\overline{x_1}(t) \to \sqrt{k_1/k_2}\,\overline{x_1}(t)$, $a_1 \to \sqrt{k_1/k_2}\,a_1$, $\tilde{x}_1 \to \sqrt{k_1/k_2}\,\tilde{x}_1$, $\overline{x_2}(t) \to k_1\sqrt{m/k_2}\,\overline{x_2}(t)$, $a_2 \to k_1\sqrt{m/k_2}\,a_2$, $\tilde{x}_2 \to k_1\sqrt{m/k_2}\,\tilde{x}_2$, $\overline{\lambda_1}(t) \to \sqrt{m/k_1}\,\overline{\lambda_1}(t)$, $\overline{\lambda_2}(t) \to m\sqrt{k_1/k_2}\,\overline{\lambda_2}(t)$, $\overline{u}(t) \to k_1/b\sqrt{k_1/k_2}\,\overline{u}(t)$, $\omega \to mb^2/k_1\omega$, and finally

$$\mathbf{R}\to\begin{bmatrix}\sqrt{m/k_2} & 0\\ 0 & m\sqrt{k_1/k_2}\end{bmatrix}\mathbf{R}\begin{bmatrix}\sqrt{k_2/k_1} & 0\\ 0 & (1/k_1)\sqrt{k_2/m}\end{bmatrix}. \tag{3.48}$$

These transformations permit rewriting Eqs. (3.45a)–(3.47) in the following form:

$$\frac{d\overline{x_1}(t)}{dt}=\overline{x_2}(t), \tag{3.49a}$$

$$\frac{d\overline{x_2}(t)}{dt}=-\overline{x_1}(t)-\overline{x_1}(t)^3+\overline{u}(t), \tag{3.49b}$$

$$\overline{x_1}(0)=a_1, \tag{3.49c}$$

$$\overline{x_2}(0)=a_2, \tag{3.49d}$$

$$\frac{d\overline{\boldsymbol{\lambda}}(t)}{dt}=-\mathbf{A}_1^T(t)\overline{\boldsymbol{\lambda}}(t)-\mathbf{Q}\overline{\mathbf{x}}(t), \tag{3.50a}$$

$$\overline{\boldsymbol{\lambda}}(t_f)=\mathbf{R}[\overline{\mathbf{x}}(t_f)-\tilde{\mathbf{x}}], \tag{3.50b}$$

$$\mathbf{A}_1(t)\equiv\begin{bmatrix}0 & 1\\ -1-3\overline{x_1}(t)^2 & 0\end{bmatrix}, \tag{3.50c}$$

$$\omega\overline{u}(t)+\overline{\lambda}_2(t)=0. \tag{3.50d}$$

In this case the kernel of the stability operator can be written as

$$\mathbf{K}(t,\tau)\equiv-\frac{1}{2\omega}[\overline{S}_\lambda^{(2)}(t,\tau)+\overline{S}_\lambda^{(2)}(\tau,t)], \tag{3.51}$$

where the superscript (2) means the second element of the sensitivity coefficient vector. To proceed, we take the functional derivative of Eqs. (3.50a) and (3.50b) with respect to $u(\tau)$.

$$\frac{\partial\overline{\mathbf{S}}_\lambda(t,\tau)}{\partial t}=-\mathbf{A}_1^T(t)\overline{\mathbf{S}}_\lambda(t,\tau)-\mathbf{Q}\overline{\mathbf{S}}_x(t,\tau), \tag{3.52a}$$

$$\overline{\mathbf{S}}_\lambda(t_f,\tau)=\mathbf{R}\overline{\mathbf{S}}_x(t_f,\tau). \tag{3.52b}$$

The solution of these equations depends on the sensitivity coefficient vector $\mathbf{S}_x(t,\tau)$ prescribed below. We now define a backward propagator $\mathbf{P}_B(t)$ as follows:

$$\frac{d\mathbf{P}_B(t)}{dt}=-\mathbf{A}_1^T(t)\mathbf{P}_B(t), \tag{3.53a}$$

$$\mathbf{P}_B(t_f)=\mathbf{I}. \tag{3.53b}$$

Then Eqs. (3.52a) and (3.52b) become

$$\overline{\mathbf{S}}_\lambda(t,\tau)=\mathbf{P}_B(t)\mathbf{R}\overline{\mathbf{S}}_x(t_f,\tau)+\int_t^{t_f}dt_1\,\mathbf{P}_B(t)\mathbf{P}_B(t_1)^{-1}\mathbf{Q}\overline{\mathbf{S}}_x(t_1,\tau). \tag{3.54}$$

To obtain an explicit expression for $\overline{\mathbf{S}}_x(t,\tau)$ we take the functional derivative of Eqs. (3.49a)–(3.49d):

$$\frac{\partial\overline{\mathbf{S}}_x(t,\tau)}{\partial t}=\mathbf{A}_1(t)\overline{\mathbf{S}}_x(t,\tau)+\delta(t-\tau)\begin{bmatrix}0\\1\end{bmatrix}, \tag{3.55a}$$

$$\overline{\mathbf{S}}_x(0,\tau)=0, \tag{3.55b}$$

which can be solved with the aid of a forward propagator $\mathbf{P}_F(t)$ satisfying

$$\frac{d\mathbf{P}_F(t)}{dt}=\mathbf{A}_1(t)\mathbf{P}_F(t), \tag{3.56a}$$

$$\mathbf{P}_F(0)=\mathbf{I}. \tag{3.56b}$$

This propagator permits a compact solution of Eq. (3.55),

$$\overline{\mathbf{S}}_x(t,\tau)=\begin{cases}0, & t<\tau\\ \mathbf{P}_F(t)\mathbf{P}_F(\tau)^{-1}\mathbf{e}_2, & t\geqslant\tau,\end{cases}$$

$$\mathbf{e}_2\equiv\begin{bmatrix}0\\1\end{bmatrix}, \tag{3.57b}$$

which leads to

$$\overline{\mathbf{S}}_\lambda(t,\tau)=\mathbf{P}_B(t)\mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F(\tau)^{-1}\mathbf{e}_2+\int_{\max(t,\tau)}^{t_f}dt_1\,\mathbf{P}_B(t)\mathbf{P}_B^{-1}(t_1)\mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F(\tau)^{-1}\mathbf{e}_2 \tag{3.58}$$

and finally yields

$$\overline{S}_\lambda^{(2)}(t,\tau)=\mathbf{e}_2^T\mathbf{P}_B(t)\mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F(\tau)^{-1}\mathbf{e}_2^T+\int_{\max(t,\tau)}^{t_f}dt_1\,\mathbf{e}_2^T\mathbf{P}_B(t)\mathbf{P}_B^{-1}(t_1)\mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F(\tau)^{-1}\mathbf{e}_2. \tag{3.59}$$

This equation can be put into a more symmetric form because the forward and backward propagators are not independent of each other. Manipulation of Eqs. (3.53a) and (3.56a) leads to

$$\frac{d}{dt}[\mathbf{P}_B^T(t)\mathbf{P}_F(t)]=\mathbf{0}, \tag{3.60}$$

which can be solved by using the conditions in Eqs. (3.53b) and (3.56) to give

$$\mathbf{P}_B^T(t)=\mathbf{P}_F(t_f)\mathbf{P}_F^{-1}(t), \tag{3.61}$$

implying that

$$\mathbf{P}_B(t)\mathbf{P}_B^{-1}(t_1)=[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t). \tag{3.62}$$

This result employed in Eq. (3.59) produces

$$\overline{S_\lambda^{(2)}}(t,\tau)=\mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_f)\mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2^T + \int_{\max(t,\tau)}^{t_f} dt_1\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T \times \mathbf{P}_F^T(t_1)\mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2, \tag{3.63}$$

which enables identifying the kernel as

$$\mathbf{K}(t,\tau)=-\frac{1}{2\omega}\Bigg[\mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_f)\mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2^T + \mathbf{e}_2^T[\mathbf{P}_F^{-1}(\tau)]^T\mathbf{P}_F^T(t_f)\mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F^{-1}(t)\mathbf{e}_2^T + \int_{\max(t,\tau)}^{t_f} dt_1\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_1)\mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2 + \int_{\max(t,\tau)}^{T} dt_1\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(\tau)]^T\mathbf{P}_F^T(t_1)\mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F^{-1}(t)\mathbf{e}_2\Bigg]. \tag{3.64}$$

Recall that the stability operator is defined by

$$\mathcal{S}\varphi(t)\equiv\int_0^{t_f} d\tau\, \mathbf{K}(t,\tau)\varphi(\tau), \tag{3.65}$$

where $\varphi(t)$ is a square integrable function over $t\in[0,t_f]$. Without loss of generality, if we assume that $\varphi(t)$ has unit norm, then the following inner product will characterize the spectrum of the stability operator:

$$(\varphi,\mathcal{S}\varphi)\equiv\int_0^{t_f}dt\int_0^{t_f}d\tau\, \varphi(t)\mathbf{K}(t,\tau)\varphi(\tau)=\sigma_1(t_f)+\sigma_2(t_f), \tag{3.66}$$

where

$$\sigma_1(t_f)\equiv-\frac{1}{\omega}\int_0^{t_f}\int_0^{t_f}dt\, d\tau\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_f) \times \mathbf{R}\mathbf{P}_F(t_f)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2^T, \tag{3.67}$$

$$\sigma_2(t_f)\equiv-\frac{1}{\omega}\int_0^{t_f}\int_0^{t_f}dt\, d\tau \times \int_{\max(t,\tau)}^{t_f} dt_1\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_1) \times \mathbf{Q}\mathbf{P}_F(t_1)\mathbf{P}_F^{-1}(\tau)\mathbf{e}_2. \tag{3.68}$$

Since $\mathbf{R}$ is a positive definite 2×2 symmetric matrix, we can write it in terms of the following spectral resolution:

$$\mathbf{R}\equiv E_L(\mathbf{R})\mathbf{r}_1\mathbf{r}_1^T+E_S(\mathbf{R})\mathbf{r}_2\mathbf{r}_2^T, \tag{3.69}$$

where $E_L(\mathbf{R})$ and $E_S(\mathbf{R})$ denote the largest and smallest eigenvalues of $\mathbf{R}$, respectively, while $\mathbf{r}_1$ and $\mathbf{r}_2$ stand for the corresponding eigenvectors. This formula permits writing $\sigma_1(t_f)$ as

$$\sigma_1(t_f)=-\frac{1}{\omega}E_L(\mathbf{R})\left(\int_0^{t_f}dt\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_f)\mathbf{r}_1\right)^2 - \frac{1}{\omega}E_S(\mathbf{R})\left(\int_0^{t_f}dt\, \mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_f)\mathbf{r}_2\right)^2, \tag{3.70}$$

which shows that $\sigma_1(t_f)$ is a negative definite function since ω is positive while $E_L(\mathbf{R})$ and $E_S(\mathbf{R})$ are non-negative.

Following along similar lines the positive definite nature and symmetry of $\mathbf{Q}$ lead to writing

$$\mathbf{Q}\equiv E_L(\mathbf{Q})\mathbf{q}_1\mathbf{q}_1^T+E_S(\mathbf{Q})\mathbf{q}_2\mathbf{q}_2^T, \tag{3.71}$$

where $E_L(\mathbf{Q})$ and $E_S(\mathbf{Q})$ denote the largest and smallest eigenvalues of $\mathbf{Q}$, respectively, while $\mathbf{q}_1$ and $\mathbf{q}_2$ stand for the corresponding eigenvectors. This formula enables expressing $\sigma_2(t_f)$ as

$$\sigma_2(t_f)=-\frac{1}{\omega}E_L(\mathbf{Q})\int_0^{t_f}dt\int_0^{t}d\tau\int_t^{t_f}dt_1\, \xi_1(t,t_1)\xi_1(\tau,t_1) - \frac{1}{\omega}E_S(\mathbf{Q})\int_0^{t_f}dt\int_0^{t}d\tau\int_t^{t_f}dt_1\, \xi_2(t,t_1)\xi_2(\tau,t_1) - \frac{1}{\omega}E_L(\mathbf{Q})\int_0^{t_f}dt\int_t^{t_f}d\tau\int_\tau^{t_f}dt_1\, \xi_1(t,t_1)\xi_1(\tau,t_1) - \frac{1}{\omega}E_S(\mathbf{Q})\int_0^{t_f}dt\int_t^{t_f}d\tau\int_\tau^{t_f}dt_1\, \xi_2(t,t_1)\xi_2(\tau,t_1), \tag{3.72}$$

where

$$\xi_1(t,t_1)\equiv\mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_1)\mathbf{q}_1, \tag{3.73a}$$

$$\xi_2(t,t_1)\equiv\mathbf{e}_2^T[\mathbf{P}_F^{-1}(t)]^T\mathbf{P}_F^T(t_1)\mathbf{q}_2. \tag{3.73b}$$

The third and fourth integrals are, respectively, the same as the first and second ones in Eq. (3.72). This comes from the following triangle identity which is valid for any arbitrary function $\phi(t,\tau)$:

$$\int_0^{t_f}dt\int_t^{t_f}d\tau\, \phi(t,\tau)\equiv\int_0^{t_f}d\tau\int_0^{\tau}dt\, \phi(t,\tau). \tag{3.74}$$

Therefore, we have

$$\sigma_2(t_f)=-\frac{2}{\omega}E_L(\mathbf{Q})\int_0^{t_f}dt\int_0^{t}d\tau\int_t^{t_f}dt_1\, \xi_1(t,t_1)\xi_1(\tau,t_1) - \frac{2}{\omega}E_S(\mathbf{Q})\int_0^{t_f}dt\int_0^{t}d\tau\int_t^{t_f}dt_1\, \xi_2(t,t_1)\xi_2(\tau,t_1). \tag{3.75}$$

If we now apply the above triangle identity to the integrations over t and t_1, then we obtain

$$\sigma_2(t_f) = -\frac{2}{\omega} E_L(\mathbf{Q}) \int_0^{t_f} dt_1 \int_0^{t_1} dt \int_0^{t} d\tau\, \xi_1(t,t_1)\xi_1(\tau,t_1) - \frac{2}{\omega} E_S(\mathbf{Q}) \int_0^{t_f} dt_1 \int_0^{t_1} dt \int_0^{t} d\tau\, \xi_2(t,t_1)\xi_2(\tau,t_1), \tag{3.76}$$

which yields

$$\sigma_2(t_f) = -\frac{1}{\omega} E_L(\mathbf{Q}) \int_0^{t_f} dt \left(\int_0^{t} d\tau\, \xi_1(\tau,t) \right)^2 - \frac{1}{\omega} E_S(\mathbf{Q}) \int_0^{t_f} dt \int_0^{t} d\tau \left(\int_0^{t} d\tau\, \xi_2(\tau,t) \right)^2. \tag{3.77}$$

Equation (3.77) implies that $\sigma_2(t_f)$ is a negative definite decreasing function of t_f. This result and the previous one for $\sigma_1(t_f)$ lead to the following conclusions.

(1) The control of a classical quartic anharmonic oscillator always exhibits stable behavior as shown by the negative definiteness of the stability operator. In the same vein the control does *not* show robustness to disturbances. Thus as t_f increases the control dynamics becomes more stable but less robust.

(2) If the control time t_f tends to increase, then the stability operator tends to be more negative definite since $\sigma_2(t_f)$ decreases, unless a special relation exists between ω and $\mathbf{Q}$. Thus as t_f increases the control dynamics becomes more stable, but less robust.

IV. CONCLUDING REMARKS

This paper presents a general framework for analyzing the optimality and robustness of any particular control solution to a deterministic system. The concepts and tools should be broadly applicable when considering the control of dynamical systems [11–13]. The perspective we follow here is distinct and complementary to that in the prior literature [6–8]. It is shown that both of these issues are dictated by the eigenvalue spectrum of the operator $\mathcal{S}$ whose kernel $\mathbf{K}(t,\tau)$ is related to the dynamically constrained functional derivative $\overline{\delta\mathbf{u}(t)}/\overline{\delta\mathbf{u}(\tau)}$ for $t_0<t,\ \tau<t_f$. No attempt was made to conduct a full functional analysis of this problem in this present paper. Here, a bound on the spectrum for linear control systems led to an interesting set of qualitative conditions regarding robustness and optimality. These conditions may serve to qualitatively guide future robust design efforts for the control of deterministic systems.

The two illustrations based on linear systems and a quartic anharmonic oscillator demonstrate different models of stability and robustness analysis. In the case of linear systems we constructed bounds to determine the spectral range of the stability operator, whereas for the quartic anharmonic oscillator case the analysis was based on demonstrating the negative definite nature of the stability operator. Each future application will have its own features for analysis, including the need for a full numerical analysis of the kernel in some cases.

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